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Spectral Methods for Partial Differential Equations

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CONTENTS

APPLICATIONS OF SPECTRAL METHODS TO TURBULENT MAGNETOFLUIDS IN SPACE AND FUSION RESEARCH

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Abstract. Recent and potential applications of spectral method computation to incompressible, dissipative magnetohydrodynamics are surveyed. Linear stability problems for one-dimensional, quasi-equilibria are approachable through a close analogue of the Orr-Sommerfeld equation. It is likely that for Reynolds-like numbers above certain as-yet-undetermined thresholds, all magnetofluids are turbulent. Four recent effects in MHD turbulence are remarked upon, as they have displayed themselves in spectral method computations: (1) inverse cascades; (2) small-scale intermittent dissipative structures; (3) selective decays of ideal global invariants relative to each other; and (4) anisotropy induced by a mean dc magnetic field. Two more conjectured applications are suggested. All the turbulent processes discussed are sometimes involved in current-carrying confined fusion magnetoplasmas and in space plasmas.

1. Introduction. Spectral method computation has had less impact on plasma physics and magnetohydrodynamics (hereafter: MHD) than it has on fluid mechanics. This is so despite the fact that there are an abundance of plasma problems which are technically more urgent, in an engineering sense, than the fluid mechanics problems to which the spectral method has typically been applied. For many of the plasma problems, there is little agreement on even the qualitative features of the solutions. It is not as much a matter of getting more accurate answers more efficiently as it is of uncovering new zeroth-order effects. This, too, is a stringent test of a numerical technique.

There has been an unfortunate gap in communication between investigators in plasma physics who must face these unsolved problems and the applied mathematicians who are in possession of the numerical expertise to deal with them. The following material is presented from the point of view of the plasma theory community and has a two-fold purpose: (1) to formalize, for spectral method numerical analysts, examples of central, unsolved problems which can be approached

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without the need to dig around in the untidy laboratory background of those problems; and (2) to remark upon some progress which has already been made by spectral techniques in the area of MHD turbulence.

Section 2 states the equations of incompressible MHD and their possible boundary conditions. They appear as a natural generalization of the Navier-Stokes equation of a viscous fluid, and define very similar classes of problems. Like the equations of a Navier-Stokes fluid, their properties will be seen to differ greatly between two- and three-dimensional flows. There is also a set of reduced MHD equations, the Strauss equations, which are intermediate between two and three dimensions and which are appropriate to the practically important case of a magnetofluid subjected to a very strong dc magnetic field.

Section 3 delineates a class of idealized MHD stability problems which parallel those related to the Orr-Sommerfeld equation for planar shear flows. No MHD problem has been studied as much as linear stability theory, but the scope and precision of the results achieved are no match for the corresponding ones for hydrodynamics.

Section 4 remarks upon some recent MHD turbulence results which have been obtained, and suggests some additional problems that seem ripe for investigation.

The need for such computations cannot be overemphasized. The plasmas of interest are often hot enough to burn up any probes which are inserted into them, and remote diagnostics are inadequate to pin down the internal MHD field variables. In many cases, all the reliable "experiments" we are going to have in the foreseeable future are likely to be those done on computers, solving what are believed to be the relevant dynamical equations. In this respect, the role computations have to play in deciding fundamental questions is potentially greater than in fluid mechanics, where the function of computers has often been in extending the accuracy of knowledge which is, in an experimental sense, nearly complete already.

2. Magnetofluid Description.

A. The MHD Equations. In this section, we write in dimensionless variables the simplest set of incompressible magnetofluid equations that could be called in some sense realistic. There are two basic solenoidal vector fields, the velocity field $\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x},t)$ and the magnetic field $\underset{\sim}{B} = \underset{\sim}{B}(\underset{\sim}{x},t)$. They obey the following four partial differential equations:

$$(1) \qquad \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\nabla\underset{\sim}{v} = \frac{1}{R}\nabla^2\underset{\sim}{v} - \nabla p + \underset{\sim}{B}\cdot\nabla\underset{\sim}{B}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{B} = \frac{1}{R_M} \nabla^2 \mathbf{B} + \mathbf{B} \cdot \nabla \mathbf{v} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0. \tag{4}$$

Taking Eqs. (1) and (2) with the last term of Eq. (1) deleted just leaves us with the dynamics of a Navier-Stokes fluid, which appears in this light as a special case of magnetofluid dynamics. p is the total (dimensionless) pressure which results from taking the divergence of Eq. (1), using Eq. (2) to eliminate the time derivative, and solving the resulting Poisson equation.

R and R_M are two dimensionless numbers whose interpretation is not unique. Two cases are of particular importance. (1) If there is as much or more fluid kinetic energy than magnetic energy, velocities may be measured in units of a mean flow speed U, lengths in units of L, a macroscopic length scale over which the fields vary, and times in units of L/U, the "eddy turnover time." In this interpretation, $R \equiv UL/\nu$, where ν is the kinematic viscosity, and is of course the Reynolds number. $R_M \equiv UL/\eta$ is the magnetic Reynolds numbers, where η is the magnetic diffusivity, $c^2/4\pi\sigma$ in cgs units, with $c \equiv$ the speed of light, and σ is the electrical conductivity. Magnetic fields, in this interpretation, are measured in units of $U\sqrt{4\pi\rho}$, cgs units, where ρ is the (uniform) mass density. They may be $\leq U$, in magnitude. (2) Sometimes the magnetic field contains an externally-imposed dc part $\mathbf{B}_0$ which is expected to remain larger than $\mathbf{v}$ and the self-consistently-supported part of $\mathbf{B}$. Then velocities may be measured in units of $C_A \equiv B_0/\sqrt{4\pi\rho}$, the Alfvén speed, and $\mathbf{B}$ may be measured in units of B_0. In this interpretation, R_M becomes $S \equiv C_A L/\eta$, the Lundquist number (often, in the fusion literature, misleadingly called the magnetic Reynolds number), and R becomes $M \equiv C_A L/\nu$, which we might call the "viscous Lundquist number." This scheme has some advantages for nearly quiescent initial conditions, since the dimensionless numbers remain constant as the mean flow speed increases. The disadvantages are that $\mathbf{v}$ and the variable part of $\mathbf{B}$ may then remain $<< 1$, and though the time scale L/U, the eddy turnover time, is still important, it may be masked by the existence of the faster L/C_A, the "Alfvén transit time," which has no direct significance for the nonlinear behavior at all.

In summary, there are four possible Reynolds-like numbers which can be of importance: $R \equiv UL/\nu$, $R_M \equiv UL/\eta$, $S \equiv C_A L/\eta$, and

$M \equiv C_A L/\nu$; when the magnetic field is a sum of an externally-imposed part plus a part which is supported by internal currents, either part may be used to define C_A. Sorting out their roles in various problems has not been fully accomplished. They always occur in front of the only dissipative terms in the equations, and in the denominators, and which pair is appropriate depends upon the problem. The situation is more involved than the Navier-Stokes case, but not hopelessly so.

B. Two-Dimensional MHD. A simplified but still rich limit of Eqs. (1) - (4) is the two-dimensional one, with all variables independent of one spatial coordinate (z, say). In the most frequently considered geometry, $\underset{\sim}{B} = (B_x, B_y, 0)$, $\underset{\sim}{v} = (v_x, v_y, 0)$ and $\partial/\partial z \equiv 0$. For this case we can write, $\underset{\sim}{B} = \nabla\times A_z\hat{e}_z$, $\underset{\sim}{v} = \nabla\times\psi\hat{e}_z$, in terms of a vector potential $A_z\hat{e}_z$ and a stream function ψ. A and ψ determine all other variables in the problem. The vorticity $\underset{\sim}{\omega} = \omega\hat{e}_z = -\nabla^2\psi\hat{e}_z$ and the current density $\nabla\times\underset{\sim}{B} = \underset{\sim}{j} = j\hat{e}_z = -\nabla^2 A_z\hat{e}_z$ have only single components and are also solenoidal, so that $j = j(x,y,t)$ and $\omega = \omega(x,y,t)$ only. Eqs. (1) - (4) get replaced by the much simpler pair

$$\frac{\partial A_z}{\partial t} + \underset{\sim}{v}\cdot\nabla A_z = \frac{1}{R_M}\nabla^2 A_z \tag{5}$$

$$\frac{\partial \omega}{\partial t} + \underset{\sim}{v}\cdot\nabla\omega = \underset{\sim}{B}\cdot\nabla j + \frac{1}{R}\nabla^2\omega. \tag{6}$$

The two possible sets of meanings attached to the four various possible Reynolds-like numbers R, R_M, S, and M, still apply. A uniform constant dc magnetic field $\underset{\sim}{B}_0 = B_0\hat{e}_z$ can be included in $\underset{\sim}{B}$ without altering Eqs. (5) and (6), or affecting the dynamics in any way.

C. The Strauss Equations. A case intermediate between two and three dimensions is the Strauss limit, appropriate to the situation with a uniform mean dc magnetic field $\underset{\sim}{B}_0 = B_0\hat{e}_z$, but with some spatial variation with z, and a perpendicular variable magnetic field $\underset{\sim}{B}_\perp \equiv \nabla\times A_z\hat{e}_z$. This limit can be derived systematically in the case $B_\perp \ll B_0$. The result of a lengthy derivation [14] is that Eqs. (5) and (6) are modified to read

$$\frac{\partial A_z}{\partial t} + \underset{\sim}{v}_\perp\cdot\nabla A_z = \frac{1}{R_M}\nabla_\perp^2 A_z + B_0\frac{\partial\psi}{\partial z} \tag{7}$$

$$\frac{\partial\omega}{\partial t} + \underset{\sim}{v}_\perp\cdot\nabla\omega = \frac{1}{R}\nabla_\perp^2\omega + \underset{\sim}{B}_\perp\cdot\nabla j + B_0\frac{\partial j}{\partial z}, \tag{8}$$

where $\underset{\sim}{v}_\perp \equiv \nabla\times\phi\hat{e}_z$ also has only x and y components also, and, $\omega = -\nabla_\perp^2\phi$, $j = -\nabla_\perp^2 A_z$, with $\nabla_\perp \equiv \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y}$. The last two terms of Eqs. (7) and (8) represent the three-dimensionality. All variables in Eqs. (7) and (8) are functions of x, y, z, and t. The parallel fields, v_z and B_z, are essentially transported as passive scalars and their dynamics need not be discussed when discussing Eqs. (7) and (8). They evolve according to a pair of linear equations which have coefficients involving $\underset{\sim}{v}_\perp$ and $\underset{\sim}{B}_\perp$, but they do not react back on $\underset{\sim}{B}_\perp$ and $\underset{\sim}{v}_\perp$.

D. Boundary Conditions. There are three sets of severely idealized simple boundary conditions for MHD fluids. The conventional analogues of "free-slip" ideal fluid boundary conditions are $\underset{\sim}{v}\cdot\hat{n} = 0$, $\underset{\sim}{B}\cdot\hat{n} = 0$ at a rigid, perfectly conducting wall along which the magnetofluid flows freely. Here $\hat{n}$ is the unit normal to the wall. These are "ideal MHD" boundary conditions and are expected to avoid boundary-layer effects.

The analogue of "no-slip," non-ideal, fluid boundary conditions at a rigid, perfect conductor are that $\underset{\sim}{v} = 0$, $\hat{n}\cdot\underset{\sim}{B} = 0$, and $\hat{n}\times\underset{\sim}{j} = \hat{n}\times(\nabla\times\underset{\sim}{B}) = 0$. All three components of the velocity vanish at a rigid, perfectly-conducting wall for a non-ideal magnetofluid, as do the normal component of the magnetic field and the tangential components of the current. For a rigid, perfectly-conducting wall coated with a thin insulating layer, a situation encountered in experimental practice, $\underset{\sim}{B}\cdot\hat{n} = 0$ survives but $\hat{n}\times\underset{\sim}{j}$ is unconstrained. Problems in which magnetic field lines are permitted to penetrate the wall material become very intractable, although in many interesting experiments such wall penetration does occur.

For two dimensions, ideal boundary conditions make A and ϕ constant around a closed conducting boundary, and non-ideal boundary conditions at a rigid perfect conductor add the conditions $\nabla_\perp^2 A = 0$, $\hat{n}\cdot\nabla_\perp\phi = 0$. (For the Strauss equations the same conditions apply, but A and ϕ may be functions of z.) For a non-ideal magnetofluid inside a rigid perfect conductor coated with a thin insulating layer, $\nabla_\perp\phi = 0$ on the boundary as does $\hat{n}\times\nabla_\perp A$, but $\nabla_\perp^2 A$ is unconstrained. (Again, in the Strauss case, A and ϕ may be functions of z.)

The foregoing three sets of MHD equations, then, accompanied by their three sets each of appropriate boundary conditions, define a far wider class of interesting problems than do the equations of a

Navier-Stokes fluid. They include Navier-Stokes flow as a special case. But they are not hopelessly more complicated than the Navier-Stokes equation. For many of these problems, even zeroth-order solutions have not been given, and it is fair to say that none of them have been studied with an intensity that approaches what has been brought to bear in fluid mechanics. We now turn to a consideration of two of these.

3. Stability of Quiescent Equilibria. A central concern in controlled fusion confinement research has been with the production of quiescent ($\underset{\sim}{v} \approx 0$) plasma states which will remain stably in place under the presence of (inevitable) small fluctuations. More theoretical plasma papers have been published about this topic than about anything else. However, much of the concern has been with device-specific situations with numerous complications, such as: toroidal geometry; compressibility; highly non-uniform dissipation coefficients (such as ν and η, which, strictly speaking, are tensors); chemical and radiative interactions with walls; inhomogeneous boundaries; impurities; etc. Drastic and uncontrolled approximations have been necessary to get any answers at all. There has been relatively little enthusiasm in the fusion community for the kind of accurate solutions of clean, simple situations analogous to those which the Orr-Sommerfeld equation requires. Yet rather spectacular success has been achieved for the Orr-Sommerfeld problem [1] by spectral methods, and a natural generalization of them to MHD stability seems a logical direction in which to proceed. In this section, we wish to define such a problem, though no results are as yet available.

The equilibrium we consider is a static one ($\underset{\sim}{v} = 0$) with a magnetic field $\underset{\sim}{B} = B_0(y)e_x$ which points in a direction perpendicular to its direction of variation. This is the magnetic analogue of plane shear flow, and has an electric current density $\underset{\sim}{j} = j_0(y)\hat{e}_z$, where $j_0(y) = -DB_0$, and $D \equiv d/dy$. For a constant resistivity, this is not strictly a static configuration, and $B_0(y)$ will resistively decay according to $B_0(y,t) = \exp(R_M^{-1} t\nabla^2)B_0(y,0)$. We will confine attention to dynamical processes which are fast on the time scale of this resistive decay, and will freeze $B_0(y)$ at its initial value. (If there are instability thresholds as η decreases, the solution will be inaccurate in the immediate vicinity of the threshold. It should be kept in mind that the very existence of such thresholds is still conjectural, however.)

If the MHD equations are linearized about this equilibrium, it is possible to generalize Squire's theorem, which says [2] that if a three-dimensional unstable mode exists, it is possible to find equivalent two-dimensional unstable modes with dissipation coefficients ν, η which are at least as high. Thresholds for instability

may be sought by restricting attention to the two-dimensional case, with $\partial/\partial z \equiv 0$.

We put rigid parallel walls at the two planes $y = \pm 1$. The perturbed fields can be written as

$$(9)\qquad \underset{\sim}{v}^{(1)} = (u,v,w)$$

$$\underset{\sim}{B}^{(1)} = (b_x,b_y,b_z),$$

and without loss of generality, we may assme the x and t dependence of all perturbed field variables is $\sim \exp(i\alpha x - i\omega t)$, where α is an arbitrary wave number and ω is a complex eigenvalue to be determined. Discarding all products of perturbation quantities, we may manipulate the components of Eqs. (1) and (4) to yield:

$$(10)\qquad (D^2-\alpha^2)^2 v = -i\omega R(D^2-\alpha^2)v - i\alpha RB_0(D^2-\alpha^2)b_y + iR\alpha(D^2B_0)b_y$$

and

$$(11)\qquad (D^2-\alpha^2 + i\omega R_M)b_y = -i\alpha B_0 R_M v,$$

which are the analogues of Orr-Sommerfeld. The boundary conditions are that $v = 0$ at $y = \pm 1$, $Dv = 0$ at $y = \pm 1$, and $b_y = 0$ at $y = \pm 1$, if the infinite parallel planes are regarded as perfectly conducting. For the sake of comparison, Orr-Sommerfeld is, in [2] the same notation, $(D^2-\alpha^2)^2 v = i\alpha R[(U_0-\omega/\alpha)(D^2-\alpha^2)v - (D^2U_0)v]$, where $U_0 = U_0(y)$ is a shear flow velocity field in the x direction.

The non-viscous $(R = \infty)$ limit of Eqs. (10) and (11) has been studied to some extent [3]. As far as I am aware, they have never even been written down for finite R and R_M (which since zero streaming has been assumed for the equilibrium, are properly interpreted as M and S). Since most plasmas of current interest are more viscous than they are resistive, it would have made more sense to have studied the $R_M = \infty$ limit. We suspect that all eigenfunctions of the system (10), (11) are stable ($\mathrm{Im}\,\omega_i < 0$) for small enough R, R_M, and that they become unstable above certain thresholds as R, R_M are increased. Just what combination of R, R_M is important, we simply do not know. Russell Dahlburg [private communication] has shown that the ideal $(M \to \infty,\ S \to \infty)$ limit of Eqs. (10) and (11) is always stable. It is tempting to speculate that, because of the extent to which the equations of motion tie

$\underset{\sim}{v}^{(1)}$ and $\underset{\sim}{B}^{(1)}$ together, the stability threshold will be determined by the smaller of M and S. For plasmas which are more viscous than resistive, this will be M. But this is pure conjecture, as are the values the critical M or S will have (will they be of the order of the Reynolds numbers for shear flow instability, a few thousand?). It is a problem for which the spectral method seems made to order.

Once questions of linear stability are settled there is again the possibility that the true stability boundary will be determined by nonlinear and three-dimensional effects, as in the case of plane shear flows (cf. Orszag & Kells [4] and Orszag & Patera [5]) and those problems, too, are of a kind for which spectral method dynamical computation has been shown to be useful.

The magnetic field profiles $B_0(y)$ that might be investigated first are: (1) B_0 constant (this is analytically soluble and the results are simply damped Alfvén waves); (2) B_0 linear in y, or j_0 = const., the uniform current profile, which would be ruled out by imposing the boundary condition $\underset{\sim}{j}\times\hat{n} = 0$ at $y = \pm 1$; (3) B_0 parabolic, j_0 linear in y, which would be again ruled out by putting perfect conductors at $y = \pm 1$; and (4) B_0 cubic, j_0 parabolic and vanishing at $y = \pm 1$. This fourth case may be thought of as roughly analogous to plane Poiseuille flow, and mocks up most closely pinch effect magnetofluids used in a generation of confinement experiments.

Still a wider class of problems results if a uniform constant, external, magnetic field is imposed parallel to the planes but at an angle to $B_0(y)\hat{e}_x$. For these, it would seem that C_A ought to be computed in terms of the internally supported part of the magnetic field. We will not take up here the class of problems which results from the assumption that there is a non-zero flow in the equilibrium.

4. Magnetohydrodynamic Turbulence. Just as Eqs. (1) - (4) may be thought of as defining a stability problem (Eqs. (10), (11) plus boundary conditions), so they also define problems of transition and fully-developed turbulence, once those thresholds are significantly exceeded. The phenomena will be more various than Navier-Stokes phenomena because of the interplay of the magnetic field with the velocity field. They are also largely undiagnosed experimentally, partially for reasons already given: internal probes of the MHD variables cannot withstand the temperatures in many plasmas of current interest. A second, and unfortunate, reason is simple bureaucratic incomprehension of the fact that many of the effects which are troublesome at high temperatures could be explored on devices which are not hot enough to burn up probes. But in the thirty-year-plus

"crash program" to produce controlled fusion, there has been deemed to be insufficient time to explore these central, basic phenomena! Computers, are, and are likely to be for some time to come, the "experimental" arbiters of any theories of MHD turbulence we may construct.

In the following pages, we will remark upon four classes of turbulent MHD processes upon which some progress using spectral method computation has already been reported. These are:

(1) Inverse Cascade Processes
(2) Small Scale Dissipative Structures
(3) Selective Decay Processes
(4) Mean Magnetic Field Anisotropy.

Lengthy computational papers have been published on all of the above topics, and the reader is referred to them for details and graphs. In the following summary remarks, there will be no attempt to reproduce representative graphical documentation for the assertions, but the relevant references will be cited.

In addition to the above summaries, two probems will be remarked upon which appear at this moment to be ripe for numerical exploration by spectral methods. These are:

(5) Turbulence in Current-Carrying Magnetofluids and
(6) Current Boundary Layers.

It is to be stressed that the computations which have been reported have seldom, if ever, involved state-of-the-art numerical methods. The affordable spatial resolution has often been pathetically small. The spectral codes have been used as an off-the-shelf physics research tool, rather than with any deep concern for the place they occupy in applied mathematics. There is not one problem on the list that could not, or should not, be done better by knowledgeable applied mathematicians.

(1) <u>Inverse Cascades</u> [6] - [14]. It is well known that under some circumstances, small scale two-dimensional Navier-Stokes turbulence may feed the large scales, somewhat reversing the usual three-dimensional picture. Precise theory for this effect is in short supply, but extensive numerical computations have given repeated verification of its existence. Generally what are required are a small scale source of excitations and at least two global ideal invariants which emphasize differently the differing parts of wave number space. The latter requirement ensures that in the part of wave number space which is approximately conservative, spectral transfer in either direction in wave number space must be accompanied by a comparable amount in the opposite direction. What is basically a statistical-mechanical constraint on the allowed phase space guarantees that any band-limited stirring mechanism must send out <u>something</u> to both longer and shorter scales.

For three-dimensional MHD turbulence, the ideal invariants are

$$\begin{aligned} \varepsilon &= \tfrac{1}{2}\int(\underset{\sim}{v}^2+\underset{\sim}{B}^2)d^3x \\ H_M &= \tfrac{1}{2}\int \underset{\sim}{A}\cdot\underset{\sim}{B}d^3x \qquad (12) \\ H_C &= \tfrac{1}{2}\int \underset{\sim}{v}\cdot\underset{\sim}{B}d^3x, \end{aligned}$$

in the presence of either periodic or perfectly-conducting boundary conditions, where $\underset{\sim}{B} = \nabla\times\underset{\sim}{A}$, in terms of the vector potential $\underset{\sim}{A}$. For the same boundary conditions, the two-dimensional ideal invariants are

$$\begin{aligned} \varepsilon &= \tfrac{1}{2}\int(\underset{\sim}{v}^2+\underset{\sim}{B}^2)d^2x = \tfrac{1}{2}\int\left[(\nabla A_z)^2 + (\nabla\phi)^2\right]d^2x \\ H_c &= \tfrac{1}{2}\int \underset{\sim}{v}\cdot\underset{\sim}{B}d^2x = \tfrac{1}{2}\int\nabla\phi\cdot\nabla A_z d^2x \qquad (13) \\ A &= \tfrac{1}{2}\int A_z^2 d^2x, \end{aligned}$$

where the notations are the same as in Section 2B. For the Strauss equations of Section 2C, the energy invariant ε and the cross-helicity H_c are still ideal invariants, and the magnetic helicity H_M degenerates into $H_M = B_0\int A_z d^3x$.

For reasons that have been gone into in considerable detail elsewhere, in all three sets of equations, there appears to be a tendency for the magnetic quantities to accumulate, in forced situations, at the longest wavelengths. The effect has been clearly demonstrated in two- and three-dimensions, but at the time of this writing is still conjectural for the Strauss equations. It is a way of accounting for large-scale magnetic fields' spontaneous generation, as in the classic "dynamo problem." This is not the only situation where there is a need to account for the occurence of large-scale magnetic activity in a system in which small-scale turbulence is present; it may be, for example, that large-scale disruptions in tokamaks can be represented as similar to inverse cascades within the framework of the Strauss equations.

(2) Small-scale Dissipative Structures [15] - [17]. Navier-Stokes turbulence has been geometrically very elusive as far as forming a concrete, intuitive picture of it is concerned. Some success at statistical characterization of turbulent Navier-Stokes flows has

been achieved, but visualizing anything explicit about what is going on has proved frustrating. Despite its primitive state, MHD turbulence has pulled ahead in one department, at least, in providing a readily visualizable description for structures associated with dissipation. The picture is clearest in two dimensions, and may not be significantly different in three. A current filament concentrates itself near a magnetic X-point, and an associated flow pattern develops which sweeps magnetic flux symmetrically toward the X point where it is dissipated. There is outflow from the weak-field corners of the X point, sometimes at a significant fraction of the Alfvén speed, and the vorticity distribution is essentially that of a quadrupole. There is a strong asymmetry between the small-scale behavior of the magnetic and mechanical quantities. The general dissipative pattern when highly disordered initial conditions are present seems to be a random collection of such current filaments and associated X-points, which for decay situations gradually become fewer over time. One important unexplored question is that of the ratio of ohmic to viscous dissipation at high Reynolds numbers, and how this ratio varies with magnetic Prandtl number, ν/η.

(3) Selective Decay Processes [18], [19]. Not entirely unrelated to inverse cascades are the decay processes which are characterized by a more rapid decay of one global quantity than another. Geometrical constraints sometimes impose lower bounds to the ratio of the more rapidly to less rapidly decaying quantity, so that as time goes on, this ratio may approach its mimimum value. The complicated evolving turbulence heads, apparently, for a simple state which can be described by a variational principle. For instance, the minimum ε/A is achieved by Eqs. (5) and (6) whenever $\underset{\sim}{v} = 0$ and $\nabla^2 A_z + k^2 A_z = 0$, where k^2 is a positive constant. The minimum $|\varepsilon/H_M|$ is achieved by Eqs. (1) - (4) when $\nabla\times\underset{\sim}{B} = \lambda\underset{\sim}{B}$, and $\underset{\sim}{v} = 0$, the "force-free" state. There is evidence both experimental and theoretical that such "selective decays" may indeed occur. The typical attempt to test the hypothesis has involved following the evolution of a set of disordered initial conditions. The global ideal invariants are all observed to decay, but at unequal rates. In a typical two-dimensional MHD run [18], ε/A was observed to decay by more than a factor of three. In a three-dimensional situation, $|\varepsilon/H_M|$ approached to within about ten percent of its theoretical lower bound [19], though not by spectral method computation. In no case known to us has there been a turbulent computation in which ε/A or $|\varepsilon/H_M|$ did anything but decay.

The situation is complicated somewhat by the fact that there appears further to be an unexplained monotonic decay of $|\varepsilon/H_C|$ in two- and three-dimensions [20]. Two differing $t \to \infty$ asymptotic states for selectively decaying situations are compatible with what

is known from high-Reynolds number computations so far, and they are not consistent with each other:

a. A state with no kinetic energy ($\underset{\sim}{v} = 0$), and ε/A(2D) or $|\varepsilon/H_M|$ (3D) minimized; and

b. A perfectly aligned state ($\underset{\sim}{v} = \pm\underset{\sim}{B}$) with perfect equipartition in which ε/A (2D), $|\varepsilon/H_c|$ (3D) are minimized, subject to the constraints that $\underset{\sim}{v} = \pm\underset{\sim}{B}$, pointwise. Further, rather demanding, computation will be required to sort out these possibilities.

(4) <u>Mean Field Anisotropy</u> [21]. Most laboratory magnetofluids of interest contain an externally-imposed dc magnetic field which selects a particular direction in space at each point. This destroys any possible isotropy of the turbulence, which is one of the most useful symmetries by which a turbulent tensor description can be simplified. The few laboratory measurements of MHD turbulence that have been made to indicate one ubiquitous feature: a large ratio ($\gtrsim 10$) of parallel to perpendicular correlation lengths, indicating a strong peaking of the $\underset{\sim}{k}$ vectors in the spectrum in a direction perpendicular to the mean field, [22], [23]. This inequality is also a key assumption in the derivation of the Strauss equations previously described. An important question is whether an initally isotropic turbulent MHD spectrum has some tendency, in the presence of a strong mean dc field, to develop an anisotropic $\underset{\sim}{k}$ spectrum which is sharply peaked perpendicularly to the direction of the mean field.

In an extensive set of 2D MHD computations (with a strong mean field in the plane of variation) Shebalin [21] has answered the question in the affirmative. What seems to be involved is an inhibited spectral transfer between Fourier modes with differing values of $\underset{\sim}{k}\cdot\underset{\sim}{B}_0$. In particular, in a resonant three-wave intéraction, $|\underset{\sim}{k}\cdot\underset{\sim}{B}_0|$ cannot increase, but $|\underset{\sim}{k}\times\underset{\sim}{B}_0|$ can. An initially isotropic distribution of $\underset{\sim}{k}$'s seems to elongate itself perpendicularly to the mean $\underset{\sim}{B}_0$ until something like the Kolmogoroff dissipation wave number is reached, so that the higher the Reynolds numbers, the more pronounced the anisotropy. Not paradoxically, if the dissipation is artificially (and unphysically) set equal to zero, the spectrum eventually reisotropizes itself uniformly over the allowed Fourier modes due to higher order processes.

Now two processes are briefly remarked upon which seem ripe for investigation by spectral method computation. They are:

(5) <u>Turbulence in Current-carrying Magnetofluids</u>. Periodic boundary conditions preclude the existence of net fluxes of (say) electric current through any cross section of the region of computation. Since many of the most interesting laboratory plasmas carry

net currents, and as a consequence have non-vanishing circulations $\oint \underset{\sim}{B}\cdot d\underset{\sim}{\ell}$ around the perimeter, rectangular periodic boundary conditions stand in the way of numerical investigation of those turbulence effects (such as those for the Strauss equations involving inverse cascades) which require a net current. The expansion of the field variables in real Fourier sine series, as constrasted with complex Fourier series with terms $\sim \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})$, seems to pose no fundamental obstacles to spectral investigation of current-driven turbulence, but no results along these lines have as yet been reported.

(6) Current Boundary Layers. The boundary condition $\underset{\sim}{j}\times\hat{n} = 0$ at a conducting boundary demands that current densities which have no reason to be small in the interior of the magnetofluid must drop to zero at the boundary, reminiscent of the effect of a rough wall on a fluid velocity. The existence of a boundary layer of current gradients is to be expected near the boundary, just as a layer of vorticity is to be expected for a fluid flowing past a surface. Current gradients are widely (and probably correctly) believed to be a source of MHD activity ("tearing modes"), and it is not impossible that any confinement device with conducting interior boundaries will contain significant amounts of current-driven magnetic fluctuation associated with current-gradient boundary layers.

5. Summary. Fluid mechanics made a transition, in the wake of engineering achievements, into a sophisticated branch of applied mathematics. Magnetofluid mechanics may follow a somewhat different trajectory, in that the engineering achievements may well depend upon the prior development of a more sophisticated approach to the relevant questions of stability, transition, and turbulence than has characterized the first few decades of investigation. Spectral method computation could prove the most useful tool of all in this undertaking.

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